Monochromatic Components With Many Edges in Random Graphs

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MIT PRIMES

October 15, 2023

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October 2024 PRIMES Conference

Overview

1. Background
   - Graphs
   - Random Graphs
   - Monochromatic Components
2. Random graph case
3. High minimum degree case

Background: Graphs

Definition

A *graph* consists of a set of vertices connected by edges, which are pairs of vertices. Pairs of vertices that share an edge are *adjacent*.

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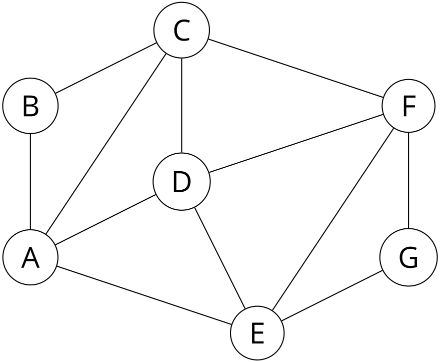

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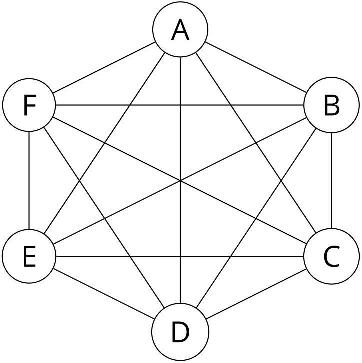
For example, K_6 is the following graph:



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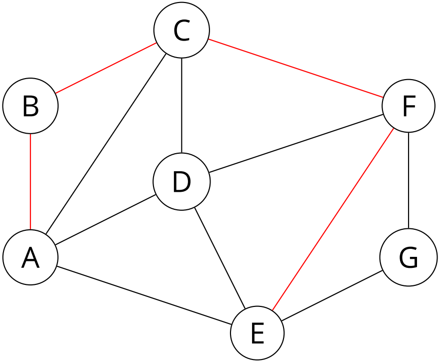
A *path* is a sequence of distinct vertices where each vertex is adjacent to the next one.



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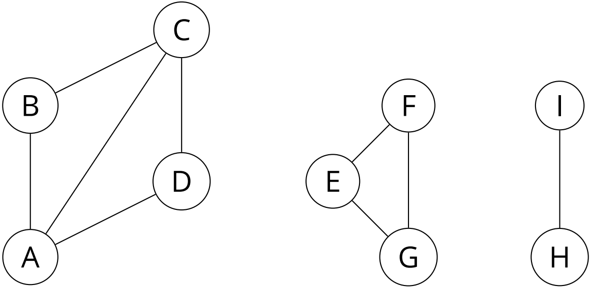

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Let n be an integer and $p \in (0, 1)$. Then, $G_{n,p}$ is a random graph on $[n]$ where each edge appears independently with probability p.

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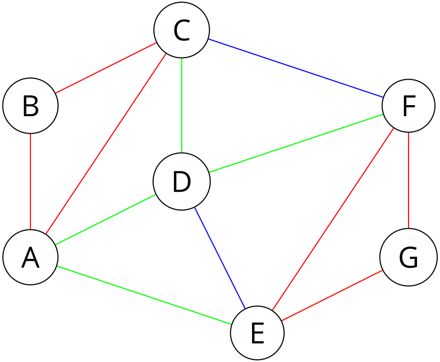
Let $\mathcal{P}$ be a graph property. Let p be a function of n. Then, $\mathcal{P}$ occurs *with high probability* on $G_{n,p}$ if

$$\lim_{n \to \infty} \mathbb{P}(\mathcal{P}(G_{n,p})) = 1.$$

Background: Monochromatic components

We can color the edges of a graph. Then, the *monochromatic connected components* are components in the graph restricted to a specific color.



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Color the edges of K_n red and blue. Then, either the red subgraph or the blue subgraph is connected.

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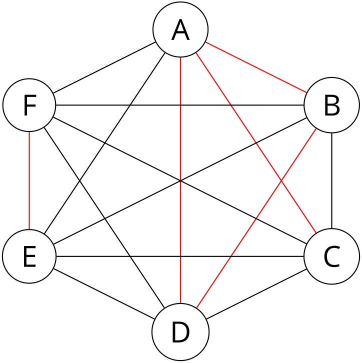

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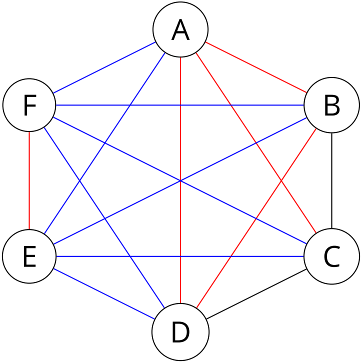

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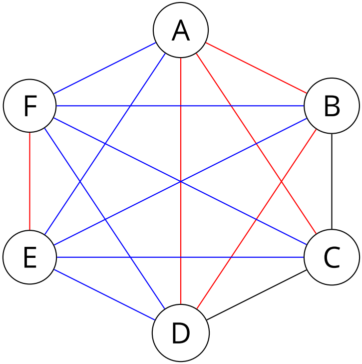

Question: Color the edges of K_n with r colors. What is the minimum possible size of the largest monochromatic connected component?

Background: Monochromatic components

Theorem (Gyárfás)

If the edges of K_n are colored with r colors, there is a monochromatic component with at least $\frac{n}{r-1}$ vertices.

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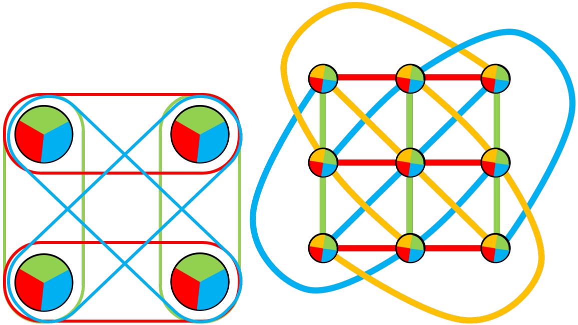

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Conjecture (Conlon, Luo, Tyomkyn)

If the edges of K_n are colored with r colors, then some monochromatic component has at least $\frac{1}{r(r-1)}\binom{n}{2}$ edges.

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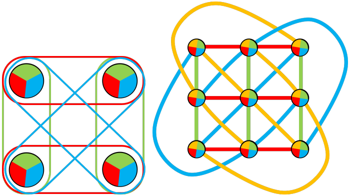

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The conjecture is true for $r = 3$ *and* $r = 4$.

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Theorem (Luo)

If the edges of K_n are colored with r colors, then some monochromatic component has at least $\frac{1}{r(r-1)+\frac{5}{4}}\binom{n}{2}$ edges.

Random graph case

Theorem (F.)

Let $p \gg \frac{\log n}{n}$. Let $G = G_{n,p}$. Let the edges of G be colored with r colors. Then, with high probability, there is a monochromatic connected component with at least $\frac{e(G)}{r^2 - r + \frac{5}{4}}(1 - o(1))$ edges.

Central idea

Theorem (Luo)

In a subgraph H of a complete r-partite graph G, there must be a connected component with at least $\frac{e(H)^2}{e(G)}$ edges.

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In a subgraph H of a complete r-partite graph G, there must be a connected component with at least $\frac{e(H)^2}{e(G)}$ *edges.*

Extension to random graphs:

- Take $G_{n,p}$ for $p = \omega(\frac{\log n}{n})$
- Lower bound the size of the subgraph
- Upper bound the size of the independent sets
- Change the bound to $(1 - o(1))\frac{e(H)^2}{e(G)}$

High minimum degree case

Theorem (F.)

Let G be a graph on n vertices where every vertex has degree at least $(1-c)n-1$, for a sufficiently small constant $c>0$. Assume the edges of G are colored red, green, and blue. Then, some color has a monochromatic connected component with at least $\frac{1}{6}e(G)$ edges.

Acknowledgements

I would like to thank my mentor, Dr. Sammy Luo, for his guidance. In addition, I would like to thank the PRIMES program for their support.

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Thanks for watching!

Questions?